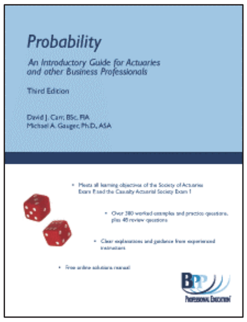

PROFESSIONAL EDUCATION

Probability, Third Edition

By David J Carr & Michael A Gauger

Published by BPP Professional Education

Solutions to practice questions – Chapter 4

Solution 4.1

(i) The sample space is:

$$S = \{BBBR, BBRB, BBRR, BRBB, BRBR, BRRB, BRRR, RBBB,$$
$$RBBR, RBRB, RBRR, RRBB, RRBR, RRRB, RRRR\}$$

(ii) The random variable X is defined as:

$$X(BBBR) = X(BBRB) = X(BRBB) = X(RBBB) = 1$$
$$X(BBRR) = X(BRBR) = X(BRRB) = X(RBBR) = X(RBRB) = X(RRBB) = 2$$
$$X(BRRR) = X(RBRR) = X(RRBR) = X(RRRB) = 3$$
$$X(RRRR) = 4$$

(iii) The random variable Y is defined as:

$$X(BBRR) = X(BRBR) = X(BRRB) = X(RBBR) = X(RBRB) = X(RRBB) = 0$$
$$X(BBBR) = X(BBRB) = X(BRBB) = X(RBBB) = X(BRRR) = X(RBRR) = X(RRBR) = X(RRRB) = 2$$
$$X(RRRR) = 4$$

Solution 4.2

The sample space is $S = \{(l, w, t): 70 < l < 120, 0 < w < 12, 0 < t < 4\}$, where l is the length of a plank, w is the width of the plank, and t is the thickness of the plank.

The random variable X is a continuous random variable defined as:

$$X(l, w, t) = lwt \text{ for all } (l, w, t) \in S$$

Solution 4.3

Since the probabilities must sum to 1, we have:

$$1 = \sum_{i=1}^{4} f_X(i) = (a)^3 + (2a)^3 + (3a)^3 + (4a)^3 = 100a^3$$

$$\Rightarrow \quad a^3 = \frac{1}{100} \quad \Rightarrow \quad a = 0.2154$$

Solution 4.4

Since all probabilities must lie between zero and one, we have:

$$0 \le \frac{1+3\theta}{4} \le 1 \quad \Rightarrow \quad -\frac{1}{3} \le \theta \le 1 \qquad\qquad 0 \le \frac{1-\theta}{4} \le 1 \quad \Rightarrow \quad -3 \le \theta \le 1$$

$$0 \le \frac{1+2\theta}{4} \le 1 \quad \Rightarrow \quad -\frac{1}{2} \le \theta \le \frac{3}{2} \qquad\qquad 0 \le \frac{1-4\theta}{4} \le 1 \quad \Rightarrow \quad -\frac{3}{4} \le \theta \le \frac{1}{4}$$

The only range which satisfies all of these is:

$$-\frac{1}{3} \le \theta \le \frac{1}{4}$$

Solution 4.5

The following table shows the product of the scores on the two dice.

		Score on die 1					
		1	**2**	**3**	**4**	**5**	**6**
Score on die 2	**1**	1	2	3	4	5	6
	2	2	4	6	8	10	12
	3	3	6	9	12	15	18
	4	4	8	12	16	20	24
	5	5	10	15	20	25	30
	6	6	12	18	24	30	36

By counting the number of (equally likely) outcomes, the probability distribution is:

$$\Pr(X=1)=\frac{1}{36} \qquad \Pr(X=8)=\frac{2}{36} \qquad \Pr(X=18)=\frac{2}{36}$$

$$\Pr(X=2)=\frac{2}{36} \qquad \Pr(X=9)=\frac{1}{36} \qquad \Pr(X=20)=\frac{2}{36}$$

$$\Pr(X=3)=\frac{2}{36} \qquad \Pr(X=10)=\frac{2}{36} \qquad \Pr(X=24)=\frac{2}{36}$$

$$\Pr(X=4)=\frac{3}{36} \qquad \Pr(X=12)=\frac{4}{36} \qquad \Pr(X=25)=\frac{1}{36}$$

$$\Pr(X=5)=\frac{2}{36} \qquad \Pr(X=15)=\frac{2}{36} \qquad \Pr(X=30)=\frac{2}{36}$$

$$\Pr(X=6)=\frac{4}{36} \qquad \Pr(X=16)=\frac{1}{36} \qquad \Pr(X=36)=\frac{1}{36}$$

Solution 4.6

Let N be the random number of claims filed in the 3-year period. Then N is a discrete random variable with possible values $0,1,2,\cdots$. The first step is to compute the probability function $p_n = \Pr(N=n)$.

The recursive relation $p_{n+1} = 0.2 \times p_n$ leads to the following:

$$p_n = 0.2 \times p_{n-1} = 0.2^2 \times p_{n-2} = \cdots = 0.2^n \times p_0$$

But since all probabilities must add to 1:

$$1 = p_0 + p_1 + \cdots = p_0\left(1 + 0.2 + 0.2^2 + \cdots\right) = p_0 \times \frac{1}{1-0.2} \quad \Rightarrow \quad p_0 = 0.8 \qquad (\text{geometric series})$$

The general formula for the probability function is thus:

$$\Pr(N=n) = p_n = 0.2^n \times p_0 = 0.2^n \times 0.8 \qquad n = 0, 1, \cdots$$

So the probability of more than 1 claim is:

$$\Pr(N>1) = p_2 + p_3 + \cdots = 1 - p_0 - p_1 = 1 - (0.8) - (0.2 \times 0.8) = 0.04$$

Solution 4.7

The cdf of X is given by:

$$F(x) = \Pr(X \le x) = \begin{cases} 0 & x < 1 \\ 0.01 & 1 \le x < 2 \\ 0.09 & 2 \le x < 3 \\ 0.36 & 3 \le x < 4 \\ 1 & 4 \le x \end{cases}$$

Solution 4.8

The pdf of X must integrate to 1. Hence:

$$1 = \int_0^\infty f(x)\,dx = \int_0^\infty 1.4e^{-kx}\,dx = \left(-\frac{1.4e^{-kx}}{k}\right)\Bigg|_0^\infty = \frac{1.4}{k}$$

$$\Rightarrow k = 1.4$$

Solution 4.9

For $x > 0$, the cumulative distribution function of X is:

$$F(x) = \Pr\left(X \le x\right) = \int_0^x f(s)ds = \int_0^x \frac{3,000}{(s+10)^4}ds = \left(-\frac{1,000}{(s+10)^3}\right)\Bigg|_0^x = 1 - \frac{1,000}{(x+10)^3}$$

The cumulative distribution function of X is fully defined as:

$$F(x) = \begin{cases} 0 & x \le 0 \\ 1 - \dfrac{1,000}{(x+10)^3} & x > 0 \end{cases}$$

Solution 4.10

The required probability is:

$$F(10) - F(4) = \left(1 - \frac{1,000}{(10+10)^3}\right) - \left(1 - \frac{1,000}{(4+10)^3}\right) = 1,000\left(\frac{1}{14^3} - \frac{1}{20^3}\right) = 0.2394$$

Solution 4.11

The required probability is:

$$\Pr\left(V > 40,000 \middle| V > 10,000\right) = \Pr\left(Y > 0.4 \middle| Y > 0.1\right) = \frac{\Pr\left(Y > 0.4 \cap Y > 0.1\right)}{\Pr\left(Y > 0.1\right)} = \frac{\Pr\left(Y > 0.4\right)}{\Pr\left(Y > 0.1\right)}$$

For $0 < y < 1$ we have:

$$\Pr\left(Y > y\right) = \int_y^1 f(t)\,dt = \int_y^1 k(1-t)^4\,dt = -\frac{k}{5}(1-t)^5\Bigg|_y^1 = \frac{k}{5}(1-y)^5$$

Finally:

$$\frac{\Pr\left(Y > 0.4\right)}{\Pr\left(Y > 0.1\right)} = \frac{(k/5)(1-0.4)^5}{(k/5)(1-0.1)^5} = \frac{(1-0.4)^5}{(1-0.1)^5} = 0.1317$$

Solution 4.12

Let Y be the amount payable from the insurance policy from a single loss.

Then Y has a mixed distribution with two discrete components:

$$\Pr\left(Y = 0\right) = \Pr\left(X \le 10\right) = \int_0^{10} 0.02e^{-0.02x}dx = \left(-e^{-0.02x}\right)\Big|_0^{10} = 1 - e^{-0.2} = 0.1813$$

$$\Pr\left(Y = 100\right) = \Pr\left(X > 100\right) = \int_{100}^{\infty} 0.02e^{-0.02x}dx = \left(-e^{-0.02x}\right)\Big|_{100}^{\infty} = e^{-2} = 0.1353$$

and a continuous component:

$$f_Y(y) = 0.02e^{-0.02y} \qquad \text{for } 10 < y < 100$$

Solution 4.13

Let X be the loss, and let Y be the insurance company's payment.

To find k, note that the probabilities must sum to 1:

$$1 = \frac{k}{2} + \frac{k}{3} + \frac{k}{4} + \frac{k}{5} + \frac{k}{6} = \frac{29k}{20} \quad \Rightarrow \quad k = \frac{20}{29}$$

The distribution of Y is related to the distribution of X as follows:

$$Y = \begin{cases} 0 & X = 1 \text{ or } 2 \\ 1 & X = 3 \\ 2 & X = 4 \\ 3 & X = 5 \end{cases}$$

So, the expected value of Y is:

$$\begin{aligned} E[Y] &= \sum y \Pr(Y = y) \\ &= 0 \times \Pr(Y = 0) + 1 \times \Pr(Y = 1) + 2 \times \Pr(Y = 2) + 3 \times \Pr(Y = 3) \\ &= \Pr(Y = 1) + 2 \times \Pr(Y = 2) + 3 \times \Pr(Y = 3) \\ &= \Pr(X = 3) + 2 \times \Pr(X = 4) + 3 \times \Pr(X = 5) \\ &= \left(\frac{20}{29} \times \frac{1}{4}\right) + 2 \times \left(\frac{20}{29} \times \frac{1}{5}\right) + 3 \times \left(\frac{20}{29} \times \frac{1}{6}\right) = 0.7931 \end{aligned}$$

Solution 4.14

Let X be the number of accidents at the factory in a particular month. Then:

$$\begin{aligned} E[X] &= \sum_{x=0}^{5} x \Pr(X = x) \\ &= (0)(0.12) + (1)(0.31) + (2)(0.26) + (3)(0.16) + (4)(0.11) + (5)(0.04) \\ &= 1.95 \end{aligned}$$

Solution 4.15

The probability density function can be written as:

$$f(x) = c(1 + x)^{-4}$$

where c is a constant of proportionality.

We can calculate the value of c using the fact that the area under the density curve is equal to 1:

$$1 = \int_0^{\infty} c(1 + x)^{-4}\, dx = -\tfrac{1}{3} c (1 + x)^{-3} \Big|_0^{\infty} = -0 - \left(-\tfrac{1}{3} c\right)$$

$$\Rightarrow \quad c = 3$$

Hence the probability density function is:

$$f(x) = 3(1+x)^{-4}$$

and the expected value is computed is:

$$E[X] = \int_0^\infty x\, f(x)\, dx = \int_0^\infty \frac{3x}{(1+x)^4} dx \qquad (\text{substitute } u = 1+x)$$

$$= \int_1^\infty \frac{3(u-1)}{u^4} du = \int_1^\infty \left(3u^{-3} - 3u^{-4}\right) du$$

$$= \left(-\tfrac{3}{2}u^{-2} + u^{-3}\right)\Big|_1^\infty = (-0+0) - \left(-\tfrac{3}{2}+1\right)$$

$$= 0.5$$

Solution 4.16

The median is the value $y_{0.5}$ such that:

$$F(y_{0.5}) = \Pr(Y \le y_{0.5}) = 0.5$$

For $y < 100$, we have:

$$\Pr(Y \le y) = \Pr(Y = 0) + \int_{10}^{y} 0.02e^{-0.02x} dx$$

$$= 0.1813 + \left(-e^{-0.02x}\right)\Big|_{10}^{y} = 0.1813 + e^{-0.2} - e^{-0.02y} = 1 - e^{-0.02y}$$

$$\Rightarrow \quad 0.5 = \Pr(Y \le y_{0.5}) = 1 - e^{-0.02y_{0.5}}$$

$$\Rightarrow \quad y_{0.5} = 34.6574$$

Solution 4.17

We have:

$$E[X^2] = \sum_{x=0}^{5} x^2 \Pr(X = x)$$

$$= (0^2)(0.12) + (1^2)(0.31) + (2^2)(0.26) + (3^2)(0.16) + (4^2)(0.11) + (5^2)(0.04)$$

$$= 5.55$$

Hence:

$$\text{var}(X) = E[X^2] - (E[X])^2 = 5.55 - 1.95^2 = 1.7475$$

So the standard deviation is $\sqrt{1.7475} = 1.3219$.

Solution 4.18

The variance can be calculated using:

$$\text{var}(X) = E[X^2] - (E[X])^2$$

We can calculate $E[X^2]$ as follows:

$$E[X^2] = \int_0^\infty x^2 \frac{3}{(1+x)^4} dx \qquad \text{(substitute } u = 1+x\text{)}$$

$$= \int_1^\infty \frac{3(u-1)^2}{u^4} du = 3\int_1^\infty \left(u^{-2} - 2u^{-3} + u^{-4}\right) du$$

$$= 3\left(-u^{-1} + u^{-2} - \frac{u^{-3}}{3}\right)\Bigg|_1^\infty$$

$$= 3(-0+0-0) - 3\left(-1+1-\frac{1}{3}\right) = 1$$

So, the variance is:

$$\text{var}(X) = E[X^2] - (E[X])^2 = 1 - 0.5^2 = 0.75$$

Solution 4.19

The variance of X is $10^2 = 100$.

Hence the variance of Y is:

$$\text{var}(Y) = \text{var}(5X+40) = 5^2 \text{ var}(X) = 2{,}500$$

Solution 4.20

We have:

$$E[C] = E[7 + 0.0742N] = E[7] + E[0.0742N] = 7 + 0.0742E[N] = 7 + 0.0742 \times 600 = 51.52$$
$$\text{var}(C) = \text{var}(7 + 0.0742N) = 0.0742^2 \text{ var}(N) = 0.0742^2 \times 250 = 1.37641$$

Solution 4.21

The mean absolute deviation is:

$$E\left[|X-\mu|\right] = \sum_{x=0}^{5} |x-\mu| \Pr(X=x)$$

$$= |0-1.95|(0.12) + |1-1.95|(0.31) + |2-1.95|(0.26) + |3-1.95|(0.16) + |4-1.95|(0.11) + |5-1.95|(0.04)$$

$$= (1.95)(0.12) + (0.95)(0.31) + (0.05)(0.26) + (1.05)(0.16) + (2.05)(0.11) + (3.05)(0.04)$$

$$= 1.057$$

Solution 4.22

The 90th percentile $x_{0.90}$ satisfies the relation:

$$F(x_{0.90}) = \Pr(X \le x_{0.90}) = 0.90$$

So, we must solve the following equation:

$$0.90 = F(x_{0.90}) = \int_0^{x_{0.90}} \frac{3}{(1+x)^4}\,dx = \left(-\frac{1}{(1+x)^3}\right)\Bigg|_0^{x_{0.90}} = 1 - \frac{1}{(1+x_{0.90})^3}$$

$$\Rightarrow \quad \frac{1}{(1+x_{0.90})^3} = 0.10$$

$$\Rightarrow \quad x_{0.90} = 10^{\frac{1}{3}} - 1 = 1.154$$

Solution 4.23

First we find the quartiles:

$$0.25 = F(x_{0.25}) = 1 - \frac{1{,}000}{(x_{0.25}+10)^3} \quad \Rightarrow \quad x_{0.25} = 1.0064$$

$$0.75 = F(x_{0.75}) = 1 - \frac{1{,}000}{(x_{0.75}+10)^3} \quad \Rightarrow \quad x_{0.75} = 5.8740$$

Hence the interquartile range is:

$$x_{0.75} - x_{0.25} = 5.8740 - 1.0064 = 4.8676$$

Solution 4.24

The skewness of X is:

$$\frac{E\left[(X-\mu)^3\right]}{\sigma^3}$$

We have:

$$\mu = E[X] = 0 \times 0.4 + 1 \times 0.6 = 0.6$$

$$E[X^2] = 0^2 \times 0.4 + 1^2 \times 0.6 = 0.6$$

$$\Rightarrow \quad \sigma^2 = E[X^2] - \left(E[X]\right)^2 = 0.24$$

$$E\left[(X-\mu)^3\right] = (0-0.6)^3 \times 0.4 + (1-0.6)^3 \times 0.6 = -0.048$$

Hence:

$$\frac{E\left[(X-\mu)^3\right]}{\sigma^3} = -\frac{0.048}{0.24^{\frac{3}{2}}} = -0.4082$$

Solution 4.25

We have:

$$\mu = E[X] = \int_0^{100} x\, f(x)dx = \left.\frac{0.01x^2}{2}\right|_0^{100} = 50$$

$$\sigma^2 = E\left[(X-\mu)^2\right] = \int_0^{100} (x-50)^2 f(x)dx = \left.\frac{0.01}{3}(x-50)^3\right|_0^{100} = 833.33$$

$$E\left[(X-\mu)^4\right] = \int_0^{100} (x-50)^4 f(x)dx = 0.002(x-50)^5\Big|_0^{100} = 1,250,000$$

Hence the kurtosis of X is:

$$\frac{E\left[(X-\mu)^4\right]}{\sigma^4} = \frac{1,250,000}{833.33^2} = 1.8$$

Solution 4.26

It's easiest to work from the cumulant generating function:

$$R_X(t) = \ln M_X(t) = 10\left(e^t - 1\right)$$

$$E[X] = R'_X(0) = 10$$

$$\text{var}(X) = R''_X(0) = 10$$

$$\Rightarrow \quad E[X^2] = \text{var}(X) + \left(E[X]\right)^2 = 110$$

Solution 4.27

Using the moment generating function:

$$M'(t) = \frac{(-1)(-20+200t)}{\left(1-20t+100t^2\right)^2} = \frac{20-200t}{\left(1-20t+100t^2\right)^2}$$

$$\Rightarrow E[X] = M'(0) = 20$$

Differentiating again using the quotient rule:

$$M''(t) = \frac{\left(1-20t+100t^2\right)^2(-200) - (20-200t)(2)\left(1-20t+100t^2\right)(-20+200t)}{\left(1-20t+100t^2\right)^4}$$

$$\Rightarrow M''(0) = \frac{-200+800}{1} = 600$$

Finally, we have:

$$\text{var}(X) = 600 - 20^2 = 200$$

Solution 4.28

In general, there is no way to relate $M_{XY}(t)$ to $M_X(t)$ and $M_Y(t)$, so let's work from first principles.

Since each X_i is either 0 (probability $\frac{1}{3}$) or 1 (probability $\frac{2}{3}$), it follows from the independence of the X_i that the possible values of $Y = X_1 X_2 X_3$ are 0 and 1 with respective probabilities:

$$\Pr(Y=1) = \Pr(\text{all } X_i = 1) = \left(\Pr(X=1)\right)^3 = \left(\frac{2}{3}\right)^3 = \frac{8}{27}$$

$$\Pr(Y=0) = 1 - \Pr(Y=1) = \frac{19}{27}$$

So the moment generating function of Y is:

$$M_Y(t) = E\left[e^{tY}\right] = \sum e^{ty}\Pr(Y=y)$$

$$= e^0 \Pr(Y=0) + e^t \Pr(Y=1) = \frac{19}{27} + \frac{8}{27}e^t$$

Solution 4.29

We have:

$$X = J + K + L$$

Hence:

$$M_X(t) = M_J(t)M_K(t)M_L(t) = (1-2t)^{-3}(1-2t)^{-2.5}(1-2t)^{-4.5} = (1-2t)^{-10}$$

Differentiating:

$$M'_X(t) = (-10)(1-2t)^{-11}(-2) = 20(1-2t)^{-11}$$

$$M''_X(t) = (-11)(20)(1-2t)^{-12}(-2) = 440(1-2t)^{-12}$$

$$M'''_X(t) = (-12)(440)(1-2t)^{-13}(-2) = 10,560(1-2t)^{-13}$$

Finally:

$$E\left[X^3\right] = M'''_X(0) = 10,560$$

Solution 4.30

If N is the number of claims, then the required probability is:

$$\Pr(80 < N \leq 150)$$

We can estimate this probability using the continuous random variable X along with a continuity correction:

$$\Pr(80 < N \leq 150) \approx \Pr(80.5 < X < 150.5)$$

$$= \int_{80.5}^{150.5} f(x)dx = \int_{80.5}^{150.5} 0.01e^{-0.01x}dx$$

$$= \left(-e^{-0.01x}\right)\Big|_{80.5}^{150.5} = -0.2220 - (-0.4471)$$

$$= 0.2251$$

